1. [5 marks: 2, 3]

A complex number, $z = a - i$, where a is a real number.

(a) Give $\overline{i\,z}$ in rectangular form.

$iz = i(a-i) = 1 + a\,i$ ✓
$\overline{i\,z} = 1 - a\,i$ ✓

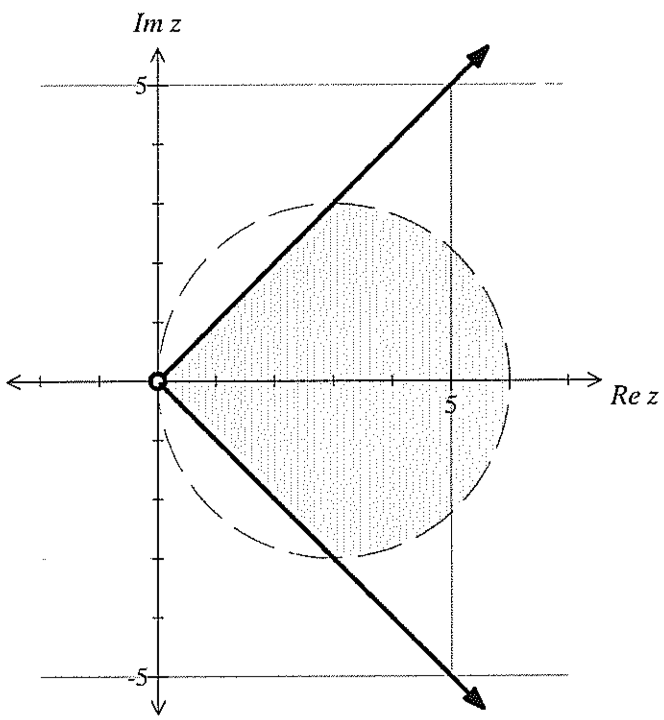
(b) Evaluate a if $z^2 = 8 + 6\,i$

$(a-i)^2 = a^2 - 2ai + i^2$
$= (a^2-1) - 2ai$ ✓
Hence:
$(a^2-1) - 2ai = 8 + 6i$ ✓
$a = -3$ ✓

2. [4 marks]

Sketch, on the complex plane provided below, the region defined by

$$|z-3| < 3 \quad \cap \quad -\frac{\pi}{4} \le \arg z \le \frac{\pi}{4}$$



✓✓✓✓

3. [9 marks: 2, 2, 5]

Find the following indefinite integrals:

(a) $\int 6\sin x\,(e^{2\cos x})\ dx$

Guess $y = e^{2\cos x}$

$\frac{dy}{dx} = -2\sin(x)\,e^{2\cos(x)}$

↓

adjust guess

$$\int 6\sin x\,(e^{2\cos x})\ dx = -3\int -2\sin x\,(e^{2\cos x})\ dx \quad ✓$$
$$= -3\,e^{2\cos x} + C. \quad ✓$$

(b) $\int \frac{7p}{5-2p^2}\ dp$

Guess $y = \ln|5-2p^2|$

$\frac{dy}{dx} = \frac{-4p}{5-2p^2}$

↓

adjust guess

$$\int \frac{7p}{5-2p^2}\ dp = \frac{7}{-4}\int \frac{-4p}{5-2p^2}\ dp \quad ✓$$
$$= \frac{7}{-4}\ln|5-2p^2| + C \quad ✓$$

(c) $\int 1-\cos^3(2x+\frac{\pi}{3})\ dx$

$$\int 1-\cos^3(2x+\frac{\pi}{3})\ dx = \int 1-\cos(2x+\frac{\pi}{3})\times\cos^2(2x+\frac{\pi}{3})\ dx$$

↓ identity

$$= \int 1-\cos(2x+\frac{\pi}{3})[1-\sin^2(2x+\frac{\pi}{3})]\ dx \quad ✓$$

expand brackets

$$= \int 1-\cos(2x+\frac{\pi}{3}) + \cos(2x+\frac{\pi}{3})\sin^2(2x+\frac{\pi}{3})]\ dx \quad ✓$$

integrate ↓

$$= x - \frac{1}{2}\sin(2x+\frac{\pi}{3}) + \frac{1}{2}\int 2\cos(2x+\frac{\pi}{3})\sin^2(2x+\frac{\pi}{3})]\ dx$$

✓ ✓

guess $y = \sin^3(2x+\frac{\pi}{3})$

$\frac{dy}{dx} = 3\sin^2(2x+\frac{\pi}{3}) \times 2\cos(2x+\frac{\pi}{3})$

↓

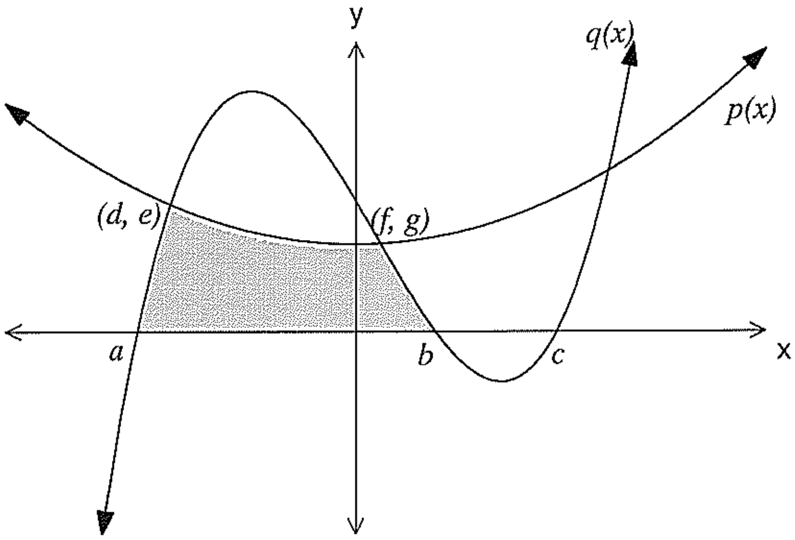
adjust guess

$$= x - \frac{1}{2}\sin(2x+\frac{\pi}{3}) + \frac{1}{6}\sin^3(2x+\frac{\pi}{3}) + C$$

✓

4. [3 marks]

A cubic function, $q(x)$, intersects the x-axis at $(a, 0)$, $(b, 0)$ and $(c, 0)$.
Another function, $p(x)$, intersects $q(x)$ in three places.
The coordinates of two of these points are shown on the graph below.



Use the information in the graph to define the shaded area in terms of definite integrals.

$$\text{Area} = \int_a^d q(x)\,dx + \int_d^f p(x)\,dx + \int_f^b q(x)\,dx$$

✓ ✓ ✓

OR

$$\text{Area} = \int_a^b q(x)\,dx - \left[\int_d^f q(x)\,dx - \int_d^f p(x)\,dx\right]$$

✓ ✓ ✓

5. [7 marks: 3, 4]

The line L has equation $\boldsymbol{r} = 4\,\mathbf{i} + 3\,\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k})$.
The plane Π has equation $\boldsymbol{r} \bullet (-\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2$.

(a) Find the position vector of the point of intersection between L and Π.

Substitute equation of L into equation of Π:

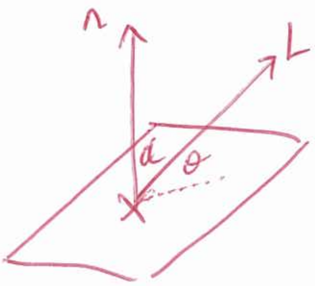
$$\begin{pmatrix} 4+\lambda \\ 3+2\lambda \\ -1-2\lambda \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 2 \qquad \checkmark$$

$$\Rightarrow \; -4 - \lambda + 3 + 2\lambda - 1 - 2\lambda = 2$$

$$\lambda = -4. \qquad \checkmark$$

Hence, point has position vector $= -5\,\mathbf{j} + 7\,\mathbf{k}$. ✓

(b) The acute angle between L and Π is θ. Find $\sin\theta$.



Angle between L and normal to plane $= (90 - \theta)°$.

$$\cos(90-\theta)° = abs\left\{ \frac{\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}\right| \bullet \left|\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right|} \right\} \qquad \checkmark$$

$$\sin\theta = abs\left\{ \frac{-1}{3 \times \sqrt{3}} \right\}$$

✓ ✓

$$= \frac{\sqrt{3}}{9}. \qquad \checkmark$$

$\cos\alpha = \frac{\sqrt{3}}{9}$

$\therefore$

$\sin\theta = \frac{\sqrt{3}}{9}$

6. [6 marks: 3, 3]

Consider a 2×2 matrix, $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$

(a) If $\mathbf{A}^2 = \alpha\mathbf{A} + \beta\mathbf{I}$ where α and β are real numbers and $\mathbf{I}$ is the 2×2 unit matrix, find α and β.

$\begin{bmatrix} 10 & 2 \\ 2 & 2 \end{bmatrix} = \alpha\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} + \beta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$10 = 3\alpha + \beta$

$2 = \alpha \quad \therefore \beta = 4$

$\mathbf{A}^2 = \begin{pmatrix} 10 & 2 \\ 2 & 2 \end{pmatrix}$ ✓

Hence $\begin{pmatrix} 10 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 3\alpha & \alpha \\ \alpha & -\alpha \end{pmatrix} + \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}$

$\Rightarrow \quad \alpha = 2 \text{ and } \beta = 4$ ✓✓

(b) Write $\mathbf{A}^4$ in the form $k\mathbf{A} + c\mathbf{I}$ where k and c are real numbers and $\mathbf{I}$ is the 2×2 unit matrix.

$\mathbf{A}^2 = 2\mathbf{A} + 4\mathbf{I}$
$= 2(\mathbf{A} + 2\mathbf{I})$

$\mathbf{A}^4 = [2(\mathbf{A} + 2\mathbf{I})]^2$ ✓
$= 4(\mathbf{A}^2 + 4\mathbf{A} + 4\mathbf{I})$
$= 4(2\mathbf{A} + 4\mathbf{I} + 4\mathbf{A} + 4\mathbf{I})$ ✓
$= 24\mathbf{A} + 32\mathbf{I}$ ✓

$A^4 = 4A^2 + 16AI + 16I^2$

$A^4 = 4[A^2 + 4AI + 4I^2]$

$A^4 = 4[(2A + 4I) + 4AI + 4I^2]$

$A^4 = 4[2A + 4I + 4A + 4I]$

$A^4 = 4[6A + 8I]$

$I^2 = I$

$4A \times I = 4A$

7. [6 marks: 2, 2, 2]

Let $u = a \text{ cis } \alpha$ and $w = b\ e^{i\beta}$ where a and b are real numbers and $-\pi < \alpha \le \pi$ and $-\pi < \beta \le \pi$.

(a) State the modulus and argument of $u \times \overline{w}$.

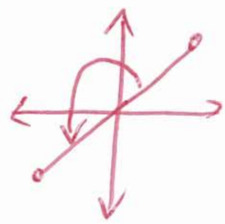
Modulus = $a \times b = ab$ ✓
Argument = $\alpha - \beta$. ✓

$[a \text{ cis } \alpha] \times [b \text{ cis } \beta]$
$= ab \text{ cis } (\alpha + \beta)$

(b) Given that u and w are the two roots of the equation $z^2 = k$, find:

(i) the relationship between a and b

$a = b$ ✓



mod doesn't change

(ii) the relationship between α and β.

$\alpha - \beta = \pm \pi$ ✓

2nd soln rotates π

(c) Given that u and w are the two roots of the equation $pz^2 + qz + r = 0$, where p, q and r are non-zero real numbers, find:

complex roots

(i) the relationship between a and b

$a = b$ ✓

(ii) the relationship between α and β.

$\alpha + \beta = 0$ ✓

quadratic formula
$a = p \quad b = q \quad c = r$

$$\frac{-q \pm \sqrt{q^2 - 4pr}}{2p}$$

$\therefore$ one soln

$$-\frac{q}{2p} + \left[\frac{\sqrt{q^2 - 4pr}}{2a}\right] i$$

2nd soln

$$-\frac{q}{2p} - \left[\frac{\sqrt{q^2 - 4pr}}{2a}\right] i$$

IN THE FORM $a + bi$

the imaginary portions are conjugates to each other

the modulus of each will be the same

8. [7 marks: 2, 3, 2]

The position vectors of the points P and Q are $-2\,\mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}$ and $6\,\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k}$ respectively.

(a) Find the position vector of K, the mid-point of the line joining P and Q.

that's fine

$$\mathbf{PQ} = \begin{pmatrix}6\\4\\5\end{pmatrix} - \begin{pmatrix}-2\\2\\-3\end{pmatrix} = \begin{pmatrix}8\\2\\8\end{pmatrix} = 2\begin{pmatrix}4\\1\\4\end{pmatrix}.$$

K is the midpoint of PQ.

Then $\mathbf{OK} = \mathbf{OP} + \frac{1}{2}\mathbf{PQ}$ ✓

$$= \begin{pmatrix}-2\\2\\-3\end{pmatrix} + \begin{pmatrix}4\\1\\4\end{pmatrix} = \begin{pmatrix}2\\3\\1\end{pmatrix}.$$ ✓

The plane Π is the perpendicular bisector of the line joining the points P and Q.

(b) Find the vector equation of the plane Π.

Required plane passes through K and is perpendicular to PQ.

Hence, a vector normal to plane = $\begin{pmatrix}4\\1\\4\end{pmatrix}$. ✓

Equation of required plane is $\boldsymbol{r} \bullet \begin{pmatrix}4\\1\\4\end{pmatrix} = \begin{pmatrix}2\\3\\1\end{pmatrix} \bullet \begin{pmatrix}4\\1\\4\end{pmatrix}$ ✓

$$\boldsymbol{r} \bullet \begin{pmatrix}4\\1\\4\end{pmatrix} = 15.$$ ✓

$r \cdot \begin{pmatrix}8\\2\\8\end{pmatrix} = 30$ ✓

(c) Find the acute angle the plane Π makes with the x-y plane.

Angle between Π and the x-y plane

= angle between $\begin{pmatrix}4\\1\\4\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$. ✓

= 45.9°. ✓

```
angle([4,1,4],[0,0,1])
                45.86825079
```

the xy plane has equation $r \cdot n = a \cdot n$ where $n = \langle 0,0,1 \rangle$ ie the normal is the z axis

the angle between the normals of two planes = the angle between the two planes.

9. [8 marks: 2, 4, 2]

A curve has equation $e^{y+x} + e^{y-x} - x^2 - 4e^y + 1 = 0$

(a) Find the exact value of the vertical intercept (y-intercept) of this curve.

$$x = 0,\ e^y + e^y - 4e^y + 1 = 0$$
$$2e^y = 1 \quad \checkmark$$
$$y = -\ln 2 \quad \checkmark$$

or $y = \ln\left(\frac{1}{2}\right)$

(b) Use an analytical method to find $\frac{dy}{dx}$.

$$e^{y+x}\left(\frac{dy}{dx} + 1\right) + e^{y-x}\left(\frac{dy}{dx} - 1\right) - 2x - 4e^y\frac{dy}{dx} = 0$$
✓ ✓ ✓

$$\frac{dy}{dx}\left(e^{y+x} + e^{y-x} - 4e^y\right) = 2x - e^{y+x} + e^{y-x}$$

$$\frac{dy}{dx} = \frac{2x - e^{y+x} + e^{y-x}}{e^{y+x} + e^{y-x} - 4e^y} \quad \checkmark$$

(c) Verify that the curve has a stationary point at its vertical intercept.

When $x = 0$, $y = \ln\left(\frac{1}{2}\right)$ ✓

$$\frac{dy}{dx} = \frac{-e^y + e^y}{e^y + e^y - 4e^y} = \frac{-e^{\ln(1/2)} + e^{\ln(1/2)}}{e^{\ln(1/2)} + e^{\ln(1/2)} - 4e^{\ln(1/2)}} = 0$$

$$\Rightarrow \frac{dy}{dx} = 0 \quad \checkmark$$

Hence, when $x = 0$, $\frac{dy}{dx} = 0$ and the curve has a stationary point.

10. [8 marks: 3, 5]

A cool room for storing food is refrigerated so that the temperature in the room, F, in degrees Celsius, at t hours after midnight, is given by the formula

$$F = -4\cos\frac{\pi(t-3)}{12} \qquad \text{for } 0 \le t \le 24$$

(a) Show that F experiences fluctuations that are similar to a particle undergoing simple harmonic motion.

$$F = -4\cos\frac{\pi(t-3)}{12}$$

$$\frac{dF}{dt} = 4\sin\frac{\pi(t-3)}{12} \times \frac{\pi}{12} = \frac{\pi}{3}\sin\frac{\pi(t-3)}{12}$$ ✓

$$\frac{d^2F}{dt^2} = \frac{\pi}{3}\cos\frac{\pi(t-3)}{12} \times \frac{\pi}{12}$$ ✓

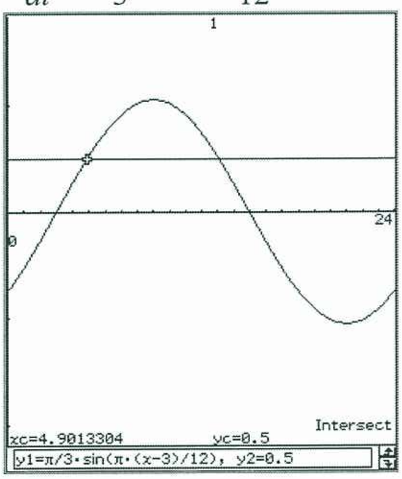
$$= \frac{\pi^2}{36}\cos\frac{\pi(t-3)}{12}$$

$$= -\left(\frac{\pi}{12}\right)^2 F$$ ✓

Which is of the form of a particle undergoing SHM.

(b) The refrigeration system automatically switches on when the rate of change of temperature, with respect to time, is greater than or equal to 0.5°C. When the rate of change of temperature, with respect to time, is less than 0.5°C per hour it automatically switches off again. Find the actual times (e.g. 2.17 a.m.), to the nearest minute, at which the system switches on and then switches off, during a 24 hour period.

$$\frac{dF}{dt} = \frac{\pi}{3}\sin\frac{\pi(t-3)}{12}$$



From graph $\frac{dF}{dt} > 0.5$

(Method clearly shown ✓)

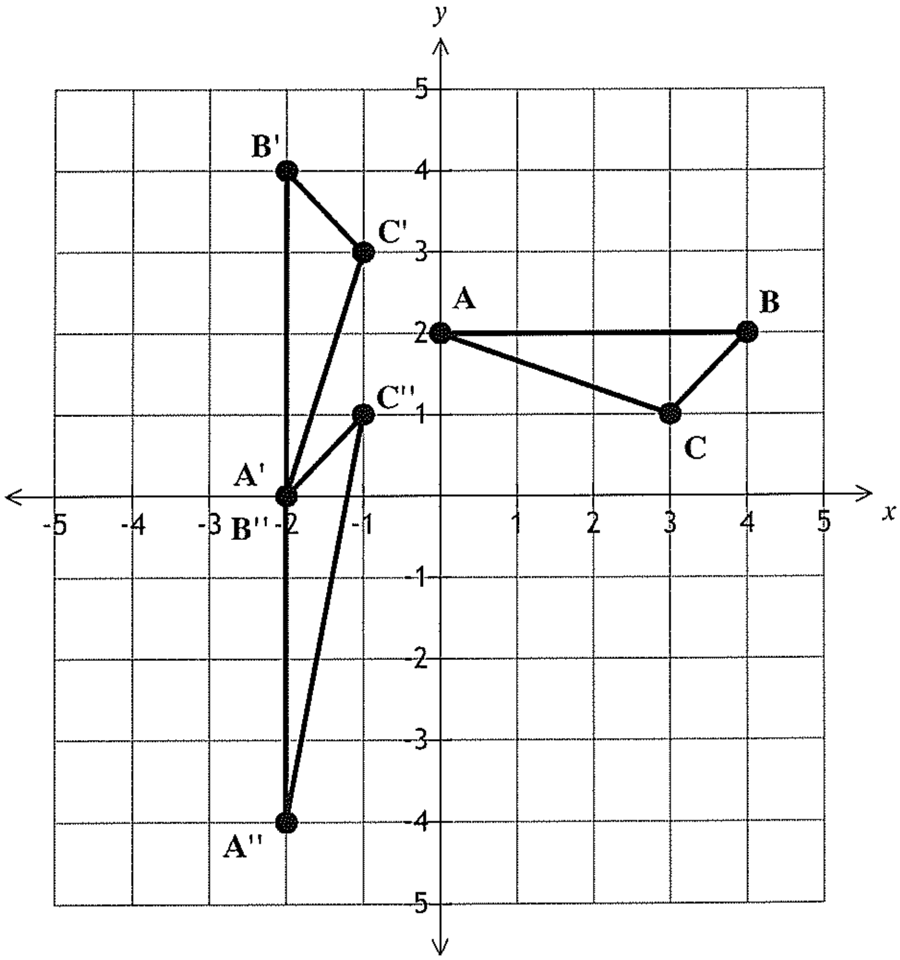
$\Rightarrow$ $4.9013 < t < 13.0987$ ✓ ✓

Hence, refrigerator turns on at 4.54 am ; turns off at 1.06 pm ✓ ✓

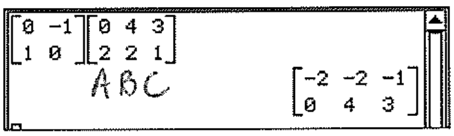
OR
Solve
$0.5 = \frac{\pi}{3}\sin\left[\frac{\pi t}{12} - \frac{\pi}{4}\right]$
$t =$

11. [7 marks: 1, 1, 2, 1, 2]

A triangle ABC is shown on the grid below with A(0, 2), B(4, 2) and C(3, 1).

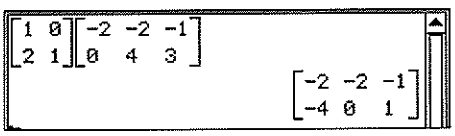


(a) On the same grid, sketch the image ΔA'B'C', after a transformation that rotates each point of the original triangle through 90° anti-clockwise about the origin.



ΔA'B'C' plotted and drawn ✓

(b) Also, sketch the image ΔA"B"C", when ΔA'B'C' is subjected to a shear transformation, of factor 2, in the *y*-direction.



Δ A"B"C" plotted and drawn ✓

11. (c) Determine the single 2 × 2 matrix that will map ΔABC directly onto ΔA"B"C".

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} \quad ✓✓$$

↑ 2nd transformation ↑ 1st transformation

(d) Find the area of each triangle drawn on the grid.

All three triangles each have area = 2 square units. ✓

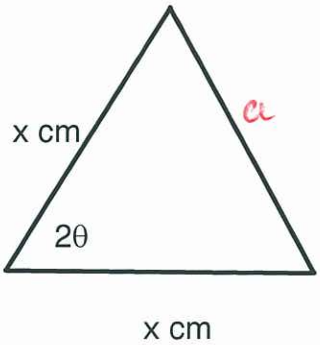
(e) The matrix $\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$, when used as a transformation matrix, will map all of the points in ΔABC onto a straight line. Give the Cartesian equation of that line.

$$\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 0 & 4 & 3 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 10 & 7 \\ 6 & 30 & 21 \end{pmatrix} \quad ✓$$

Equation of line passing through
(2, 6), (10, 30) and (7, 21) is $y = 3x$. ✓

12. [11 marks: 2, 4; 5]

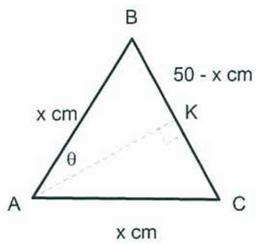
The diagram below shows an isosceles triangle with two sides both x cm and the included angle 2θ radians.



$a^2 = x^2 + x^2 - 2(x)(x)\cos 2\theta$
$a^2 = 2x^2 - 2x^2\cos 2\theta$
$a^2 = 2x^2 - 2x^2[1 - 2\sin^2\theta]$
$a^2 = 2x^2 - 2x^2 + 4x^2\sin^2\theta$
$a^2 = 4x^2\sin^2\theta$
$a = 2x\sin\theta$
$\therefore 100 = x + x + 2x\sin\theta$
$100 = 2x + 2x\sin\theta$
$50 = x + x\sin\theta$
$50 - x = x\sin\theta$
$\frac{50-x}{x} = \sin\theta$

(a) If the perimeter of the triangle is fixed at 100 cm.

(i) Prove that $\sin\theta = \dfrac{50-x}{x}$.



BC = 100 – 2x cm.
Since, triangle is isosceles, ∠ BAK = θ
and BK = $\frac{1}{2}(100 - 2x) = 50 - x$. ✓

Hence in ΔAKB, $\sin\theta = \dfrac{50-x}{x}$. ✓

(ii) Find the exact value(s) of x and θ when the area of the triangle is a maximum.

AK = $\sqrt{x^2 - (50-x)^2} = 10\sqrt{x-25}$ ✓

Hence, Area $A = \frac{1}{2} \times (100 - 2x) \times 10\sqrt{x-25}$

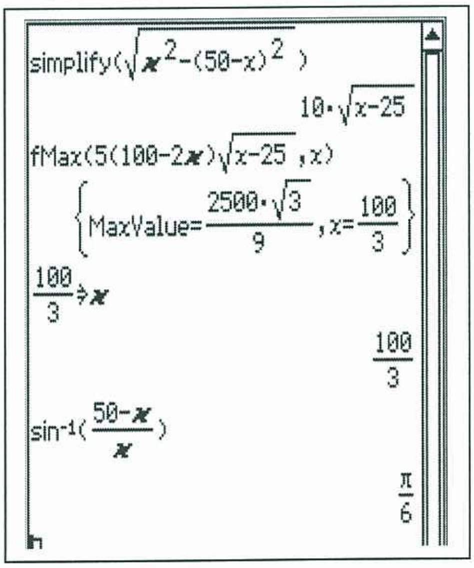
$= 5(100 - 2x)\sqrt{x-25}$ ✓

Use fMax command on CAS calculator

Hence, A is max when $x = \dfrac{100}{3}$ cm ✓

$\Rightarrow \sin\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}$. ✓

Find $\frac{dA}{dx} = 0$



12. (b) The perimeter of the triangle is no longer fixed at 100 cm.

The sides with length x cm are increasing at a constant rate of 1cm per minute.
The included angle is increasing at a constant rate of 0.1 radians per minute.
Find the exact rate at which the area of the triangle is increasing

when $x = 10$ cm and $\theta = \frac{\pi}{6}$ radians.

Let $\alpha = 2\theta$

Area $A = \frac{1}{2} \times x \times x \times \sin\alpha$

$= \frac{1}{2}x^2 \sin\alpha$ ✓

Differentiate implicitly with respect to time t:

$\frac{dA}{dt} = x\frac{dx}{dt}\sin\alpha + \frac{1}{2}x^2\cos\alpha\frac{d\alpha}{dt}.$ ✓✓

When $x = 10$, $\alpha = \frac{\pi}{3}$, $\frac{dx}{dt} = 1$, $\frac{d\alpha}{dt} = 0.1$: ✓

$\frac{dA}{dt} = 10 \times 1 \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times 100 \times \frac{1}{2} \times 0.1$

$= 5\sqrt{3} + \frac{5}{2}$ cm^2 per minute ✓

$\frac{d\theta}{dt} = 0.1$

$\frac{d(2\theta)}{dt} = 0.05$

$A = \frac{1}{2}x^2 \sin(2\theta)$

$\frac{dA}{dt} = x\frac{dx}{dt}\sin(2\theta) + \frac{1}{2}x^2\, 2\cos(2\theta)\frac{d(2\theta)}{dt}$

$\frac{dA}{dt} = x\frac{dx}{dt}\sin(2\theta) + x^2\cos(2\theta)\frac{d(2\theta)}{dt}$

$= 10(1)\sin\left(\frac{\pi}{3}\right) + 10^2\cos\left(\frac{\pi}{3}\right)[0.05]$

$= 10\frac{\sqrt{3}}{2} + 100\left(\frac{1}{2}\right)(0.05)$

13. [7 marks: 2, 2, 3]

Three people, Andrew, Benjamin, and Charles, kick a soccer ball to each other. There is a probability of $\frac{1}{4}$ that Andrew will kick the ball to Benjamin, there is a probability of $\frac{3}{5}$ that Benjamin will kick the ball to Charles and there is probability of $\frac{1}{3}$ that Charles will kick the ball to Andrew. Assume that each person does not kick the ball to himself. This information is summarized in a transition matrix

$$\mathbf{T} = \text{To}\ \begin{matrix} A \\ B \\ C \end{matrix} \overset{\text{From}}{\overset{\begin{matrix} A & B & C \end{matrix}}{\begin{pmatrix} 0 & \frac{2}{5} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{2}{3} \\ \frac{3}{4} & \frac{3}{5} & 0 \end{pmatrix}}}.$$

(a) Given that Andrew had the first kick, find the probability that Andrew will have the ball back after the ball has been kicked twice (this includes Andrew's first kick).

$$\mathbf{T}^2 = \begin{pmatrix} 0 & \frac{2}{5} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{2}{3} \\ \frac{3}{4} & \frac{3}{5} & 0 \end{pmatrix}^2 = \begin{pmatrix} \frac{7}{20} & \frac{1}{5} & \frac{4}{15} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{12} \\ \frac{3}{20} & \frac{3}{10} & \frac{13}{20} \end{pmatrix} \quad \checkmark$$

Hence required probability = $\frac{7}{20}$ ✓

(b) Given that Benjamin had the first kick, find the probability that Charles will have the ball after the ball has been kicked five times.

$$\mathbf{T}^5 = \begin{pmatrix} 0 & \frac{2}{5} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{2}{3} \\ \frac{3}{4} & \frac{3}{5} & 0 \end{pmatrix}^5 = \begin{pmatrix} \frac{11}{40} & \frac{11}{40} & \frac{61}{240} \\ \frac{17}{64} & \frac{5}{16} & \frac{19}{48} \\ \frac{147}{320} & \frac{33}{80} & \frac{7}{20} \end{pmatrix} \quad \checkmark$$

Hence required probability = $\frac{33}{80}$ ✓

13. (c) In the long term, who is most likely to end up with the ball? Justify your answer.

$$\mathbf{T}^{100} = \begin{pmatrix} 0 & \frac{2}{5} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{2}{3} \\ \frac{3}{4} & \frac{3}{5} & 0 \end{pmatrix}^{100} = \begin{pmatrix} \frac{4}{15} & \frac{4}{15} & \frac{4}{15} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}$$

✓

In the long term:
Andrew has a 0.2667 chance of ending up with the ball.
Benjamin has a 0.3333 chance of ending up with the ball.
Charles has a 0.4 chance of ending up with the ball. ✓

Hence, Charles is the most likely person to end up with the ball. ✓

14. [7 marks: 3, 2, 2]

In a chemical process, the quantity of an enzyme (Q mg) is modelled by the equation $\frac{dQ}{dt} = (200 - Q) \times t$ where t is time in hours.

(a) Use integration to find an expression for Q in terms of t.

$$\frac{dQ}{dt} = (200 - Q) \times t$$

$$\int \frac{dQ}{(200 - Q)} = \int t \; dt \quad ✓$$

$$-\ln(200 - Q) = \frac{t^2}{2} + C. \quad ✓$$

$$200 - Q = Ae^{-\frac{t^2}{2}}$$

$$Q = 200 - Ae^{-\frac{t^2}{2}} \quad ✓$$

(b) If the initial amount of the enzyme is 1000 mg, how much remains after 3 hours?

$1000 = 200 - A \quad \Rightarrow A = -800$ ✓

$Q(3) = 200 + 800e^{(-9/2)} \approx 208.9$ mg ✓

(c) Show clearly why the long term quantity of the enzyme is not dependent on its initial amount.

Since $Q = 200 - Ae^{-\frac{t^2}{2}}$

As $t \to \infty$, $Ae^{-\frac{t^2}{2}} \to 0$ and $Q \to 200$. ✓

Clearly the final amount 200 mg is independent of Q(0). ✓

15. [6 marks]

Use the substitution $x = \frac{5}{2}\sin\theta$, to evaluate exactly $\int_0^{\frac{5}{4}} \frac{1}{\sqrt{25-4x^2}}\,dx$.

Show clearly each step of your working.

$$x = \frac{5}{2}\sin\theta \quad \Rightarrow \quad dx = \frac{5}{2}\cos\theta\,d\theta \qquad ✓$$

$$x = 0 \quad \Rightarrow \quad \theta = 0$$

$$x = \frac{5}{4} \quad \Rightarrow \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{6} \qquad ✓$$

$$I = \int_0^{\frac{5}{4}} \frac{1}{\sqrt{25-4x^2}}\,dx$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{25-4\left(\frac{5\sin\theta}{2}\right)^2}} \times \frac{5\cos\theta}{2}\,d\theta \qquad ✓$$

$$= \frac{5}{2}\int_0^{\frac{\pi}{6}} \frac{\cos\theta}{\sqrt{25-\left(25\sin^2\theta\right)}}\,d\theta$$

$$= \frac{5}{2}\int_0^{\frac{\pi}{6}} \frac{\cos\theta}{5\sqrt{1-(\sin\theta)^2}}\,d\theta \qquad ✓$$

$$= \frac{5}{2}\int_0^{\frac{\pi}{6}} \frac{\cos\theta}{5\cos\theta}\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} 1\,d\theta$$

$$= \frac{1}{2}\left[\,\theta\,\right]_0^{\frac{\pi}{6}} \qquad ✓$$

$$= \frac{\pi}{12} \qquad ✓$$

16. [6 marks]

A particle P moves in the x-y plane. Its equation of motion is given by:
$\frac{dy}{dt} = 2\sin(2t)$ and $\frac{dx}{dt} = \cos(t)$, where t is time in seconds. Given that the particle P starts from the point (0, 0), find the Cartesian equation of the path traced by this particle.

$\frac{dy}{dt} = 2\sin(2t) \Rightarrow y = -\cos(2t) + A$

$t = 0, y = 0 \Rightarrow A = 1 \Rightarrow y = -\cos(2t) + 1$ ✓✓

$\frac{dx}{dt} = \cos(t) \Rightarrow x = \sin(t) + B$

$t = 0, x = 0 \Rightarrow B = 0 \Rightarrow x = \sin(t)$ ✓✓

But $y = -[1 - 2\sin^2(t)] + 1$ ✓

$= 2x^2$ ✓

$y = -\cos(2t) + 1$

$y = -[1 - 2\sin^2(t) + 1]$

$y = -[1 - 2[x]^2 + 1]$

$y = 2x^2$

17. [6 marks]

Prove that $(1 + \cos 2\theta + i \sin 2\theta)^n = 2^n \cos^n \theta\,(cis\, n\theta)$.

$$
\begin{aligned}
\text{LHS} &\equiv (1 + \cos 2\theta + i \sin 2\theta)^n \\
&\equiv (2\cos^2\theta + i \sin 2\theta)^n && \checkmark \\
&\equiv (2\cos^2\theta + i\, 2 \sin\theta \cos\theta)^n && \checkmark \\
&\equiv [2\cos\theta\,(\cos\theta + i \sin\theta)]^n && \checkmark \\
&\equiv 2^n \cos^n\theta\,(\cos\theta + i\sin\theta)^n && \checkmark \\
&\equiv 2^n \cos^n\theta\,(cis\,\theta)^n && \checkmark \\
&\equiv 2^n \cos^n\theta\,(cis\,n\theta) && \checkmark \\
&\equiv \text{RHS}
\end{aligned}
$$

$$
\begin{aligned}
\text{LHS}\quad & [1 + \cos 2\theta + i \sin 2\theta]^n \\
&= [1 + (-1 + 2\cos^2\theta) + (i\, 2\sin\theta \cos\theta)]^n \\
&= [2\cos^2\theta + i\, 2\sin\theta \cos\theta]^n \\
&= [2\cos\theta\,(\cos\theta + i\sin\theta)]^n \\
&= 2^n \cos^n\theta\,(cis\,\theta)^n \\
&= 2^n \cos^n\theta\,(cis\,n\theta)
\end{aligned}
$$

18. [7 marks]

Using mathematical induction, prove that, for all counting numbers, n,
$2n\,(2n+1)(2n-1)$ is divisible by 6.

Let $P(n) = 2n\,(2n+1)(2n-1)$
For $n = 1$: $P(1) = 2(3)(1)$
$= 6$ which is divisible by 6.
Hence, conjecture is true for $n = 1$. ✓

Assume that conjecture is true for $n = k$:
That is, $2k\,(2k+1)(2k-1)$ is divisible by 6.
$\Rightarrow$ $2k\,(2k+1)(2k-1) = 6m$ for some counting number m. ✓

For $n = k + 1$:

$$P(k+1) = 2(k+1)(2k+3)(2k+1)$$

$$= (2k+2)(2k+3)(2k+1)$$

$$= 2k\,(2k+3)(2k+1) + 2\,(2k+3)(2k+1) \quad ✓$$

$$= 2k\,(2k+1)(2k-1+4) + 2\,(2k+3)(2k+1)$$

$$= 2k\,(2k+1)(2k-1) + 4 \times 2k\,(2k+1) + 2\,(2k+3)(2k+1) \quad ✓$$

$$= 2k\,(2k+1)(2k-1) + 2\,(2k+1)[4k+2k+3]$$

$$= 2k\,(2k+1)(2k-1) + 2\,(2k+1)(6k+3) \quad ✓$$

$$= 2k\,(2k+1)(2k-1) + 6\,(2k+1)(2k+1)$$

$$= 6m + 6\,(2k+1)(2k+1). \quad ✓$$

Hence, $P(k+1)$ is divisible by 6.

Hence, if the conjecture is assumed to be true for $n = k$,
then it must be true for $n = k + 1$.

Since the conjecture is true for $n = 1$,
using the result just shown, it must then be true for $n = 1 + 1 = 2$.
Since it is true for $n = 2$, it must be true for $n = 2 + 1 = 3$.
Since it is true for $n = 3$, it must be true for $n = 3 + 1 = 4$, and so on. ✓

Hence, the result must be true for all counting numbers n.

18. [7 marks]

Using mathematical induction, prove that, for all counting numbers, n,
$2n\,(2n + 1)(2n - 1)$ is divisible by 6.

Let $n = 1$ $\quad 2(3)(1) = 6$ $\quad$ CONJECTURE TRUE FOR $n = 1$ ✓

Assume true for $n = k$

$2k(2k+1)(2k-1) = 6M$ $\quad M \in$ Natural

$8k^3 - 2k = 6m$ ✓

For $n = k+1$

$2(k+1)(2[k+1]+1)(2[k+1]-1)$

$= 2(k+1)(2k+3)(2k+1)$

$= (2k+2)(2k+3)(2k+1)$

$= 8k^3 + 24k^2 + 22k + 6$ ✓

$= 8k^2 - 2k + 24k^2 + 24k + 6$ ✓

$= 6m + [24k^2 + 24k + 6]$ ✓

$= 6m + 6[4k^2 + 4k + 1]$ ✓

$\therefore$ when $n = k+1$ it is divisible by 6.

↓ ✓

END OF PAPER